Letter

Higher-order Ising model on hypergraphs

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Non-dyadic higher-order interactions affect collective behavior in various networked dynamical systems. Here, we discuss the properties of a novel Ising model with higher-order interactions and characterize its phase transitions between the ordered and the disordered phase. By a mean-field treatment, we show that the transition is continuous when only three-body interactions are considered, but becomes abrupt when interactions of higher orders are introduced. Using a Georges-Yedidia expansion to go beyond a naïve mean-field approximation, we reveal a quantitative shift in the critical point of the phase transition, which does not affect the universality class of the model. Finally, we compare our results with traditional p-spin models with many-body interactions. Our work unveils new collective phenomena on complex interacting systems, revealing the importance of investigating higher-order systems beyond three-body interactions.

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Many complex systems are characterized by non-pairwise interactions among the system's units [1]. Taking into account the higher-order structure of networks has led to the discovery of new phenomena and collective behaviors across a wide range of dynamical processes [2], including contagion [3–6], diffusion [7–9], percolation [10–12], synchronization [13–17], evolutionary dynamics in social dilemmas [18,19], and ecology [20,21].

The Ising model is one of the simplest models to display a phase transition, universality, and complex phenomenology [22]. Originally introduced to study the order-disorder transition in lattices for ferromagnets, it has been extended to consider general interaction structures modeled as complex networks [23–25], finding a rich phenomenology where the presence and nature of the phase transition depend on the specific shape of the degree distribution. Many-body interactions have been considered before in spin models, under the name of p-spin models [26]. Yet, the straightforward extension of the Ising model to higher-order interactions provided by the p-spin model breaks the $\mathbb{Z}_2$ symmetry of the pairwise model when odd p interactions are considered and assigns the same energy to different spin configurations when there are even p interactions. Here, we study the phase transitions of a novel spin model with dyadic and group interactions (Fig. 1), which we originally proposed in Ref. [27] to overcome the limitations of the traditional p-spin approaches.

We develop a mean-field treatment of this model and discuss its emergent behaviors in the case of arbitrarily structured and heterogeneous higher-order interactions modeled by hypergraphs. Our model displays complex behavior, with the nature of the disorder-order phase transition depending on the maximum order of the many-body interactions: continuous for three-body interactions, and discontinuous for higher orders. We perform a high-temperature expansion of the free energy of our model to go beyond the mean-field approximation, showing that sparse connectivity induces a correction in the critical point marking the onset of the phase transition, without affecting the universality class of the model. We also compare our results with traditional ferromagnetic p-spin models in the literature.

Model. We consider the Ising model on higher-order networks that we introduced in Ref. [27]. This model is defined

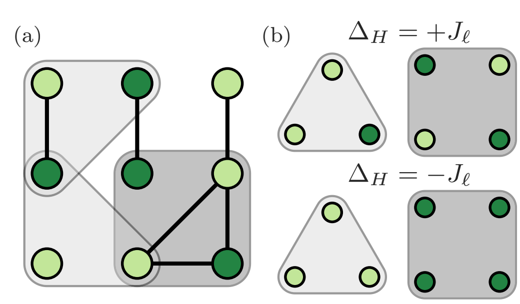

FIG. 1. Illustration of the higher-order Ising model. (a) The model is defined on a hypergraph, the nodes interact through edges and hyperedges of different orders. (b) Hyperedges of order ℓ contribute to the energy with a negative term $\Delta_\ell = -J_\ell$ if all the nodes contained in them are aligned; otherwise, the contribution has the opposite sign.

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on a hypergraph $\mathcal{H} = (V, E)$ with $|V| = N$ nodes and hyperedges $\{\sigma\} = E$—subsets of elements of V—of order (number of nodes participating in the hyperedge minus 1) up to $\ell_{\max}$. To each node $i \in V$, we associate a binary state variable $s_i \in \{\pm 1\}$, corresponding to the spin state of the node, which can either be up ($s_i = +1$) or down ($s_i = -1$). The global state of the system is controlled by the Hamiltonian:

$$H^{\mathrm{CS}} = -h \sum_i s_i - \sum_{\ell=1}^{\ell_{\max}} J_\ell \sum_{\{\sigma \in \mathcal{H} : |\sigma| = \ell\}} \left[2 \bigotimes_{i \in \sigma} s_i - 1 \right] \quad (1)$$

where

$$\bigotimes_{i=1}^{n} s_i = \delta(s_1, \dots, s_n) = \begin{cases} 1 & \text{if } s_1 = \dots = s_n \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

is the Kronecker delta for an arbitrary number of binary arguments. This higher-order generalization of the dyadic model echoes previous works extending pairwise contagion models to higher-order networks, where susceptible nodes can get infected via additional group mechanisms if all the other nodes participating in a given hyperedge are infected [3]. Note that if the spin variables take discrete values $s_i \in \{0, 1, \dots, q-1\}$, Eq. (1) can be modified to describe a generalization to arbitrary orders of interaction of the standard Pott's model [28]. An alternative extension of the Ising model to account for many-body interactions is described by the Hamiltonian:

$$H^{\mathrm{BS}} = -h \sum_i s_i - \sum_{\ell=1}^{\ell_{\max}} J_\ell \sum_{\{\sigma \in \mathcal{H} : |\sigma| = \ell\}} \prod_{i \in \sigma} s_i. \quad (3)$$

This model is commonly referred to in the literature as the p-spin model (with $p = \ell + 1$). When there are only interactions of order $\ell = 2$ and $h = 0$, it is also known as the Newman-Moore or Triangular Plaquette model [29]. It has been extensively studied across various contexts, including fully connected systems [30], diluted systems [31,32], disordered systems [26], and regular lattices [33,34]. As already noted in the first analysis of the Ising model with three-body interactions, when there are interactions between an odd number of spins, the extension provided by Eq. (3) breaks the parity symmetry (for this reason we refer to this model as *broken symmetry* model, BS, and to our model described by Eq. (1) as *conserved symmetry*, CS) under spin flip at all sites of the dyadic model without magnetic field [35]. This can be easily understood considering the energy of a system constituted by three spins connected in a 2-hyperedge. There are two states with all spins aligned: all spins pointing up and all spins pointing down. These two configurations are symmetric upon flipping all spins, yet their energies when computed using Eq. (3) are different, favoring the state with spins pointing up. Moreover, when considering orders $\ell > 2$, Eq. (3) assigns the same energy to very different spin configurations when quadratic terms are present. For example, considering a plaquette interaction of four spins, this model will assign the same (ground-state) energy to the configurations in which all the spins are aligned and to the configurations in which there are pairs of spins pointing in the same direction. The CS model was originally formulated to overcome these inherent limitations of the traditional ferromagnetic p-spin model.

Homogeneous mean-field. We develop a homogeneous mean-field treatment of the model based on two assumptions [36] First, we will write the spin state at site i as

$$s_i = \langle s_i \rangle + \Delta s_i = \langle s_i \rangle + (s_i - \langle s_i \rangle). \quad (4)$$

We simplify the coupling terms appearing in the Hamiltonians by neglecting all terms that are second-order in the fluctuations. Second, we assume that the expectation value of the spin state is uniform in the entire system:

$$\langle s_i \rangle = m \;\; \forall i \quad (5)$$

where the magnetization $m = \sum_i s_i / N$ is the order parameter of the system, taking values in the interval $[-1, +1]$. The pairwise Ising model (corresponding to our model and to the p-spin model with $\ell_{\max} = 1$) with no magnetic field, when varying the inverse temperature β, undergoes a second-order phase transition between a disordered ($m = 0$ for $\beta = 0$) and an ordered phase ($|m| = 1$ for $\beta \to \infty$). This phase transition can alternatively be expressed by fixing the inverse temperature β and varying the coupling strength J_1. Using Eqs. (4) and (5), we can write the product of two spins appearing in the coupling term of the pairwise Ising model as $m(s_i + s_j) - m^2$. In Eq. (1), we will have the same term for $\ell = 1$ as for $s_i, s_j \in \{-1, +1\}$ we have the identity:

$$2\delta(s_i, s_j) - 1 \equiv s_i s_j \iff \delta_{s_i, s_j} \equiv \frac{s_i s_j + 1}{2}. \quad (6)$$

Considering interactions of arbitrary order ℓ, we make use of the identity:

$$\bigotimes_{i=1}^{n} s_i \equiv \prod_{j=2}^{n} \delta(s_1, s_j). \quad (7)$$

We can thus write the many-body terms appearing in the Hamiltonian Eq. (1) as

$$2 \bigotimes_{i=1}^{n} s_i = \frac{1}{2^{n-1}} \left[\sum_{\alpha=2}^{2\lfloor \frac{n}{2} \rfloor} \prod_{i=1}^{\alpha} s_i + 1 \right] \quad (8)$$

where $2\lfloor \frac{n}{2} \rfloor$ is the largest even number smaller or equal to n. For instance, if $n = 5$, $2\lfloor \frac{n}{2} \rfloor = 4$ and the latter equation will contain products of spins up to size 4. Considering the product of α spins appearing in Eq. (8), in the mean-field approximation we have

$$\prod_{i=1}^{\alpha} s_i \simeq m^{\alpha-1} \sum_{i=1}^{\alpha} s_i - ((\alpha - 1)\, m^\alpha). \quad (9)$$

Inserting Eq. (9) into Eq. (8) allows us to write explicitly the decoupling of the Kronecker delta in terms of the single-nodes states and powers of the magnetization. At all orders, under the mean-field assumption, we decouple all spins, obtaining a fully decoupled Hamiltonian $H(m) = -h_{\mathrm{eff}} \sum_i s_i$. As the coupling parameters J_ℓ appear in $H(m)$ multiplied with the corresponding generalized average degree $\langle d_\ell \rangle$, we introduce the rescaled coupling parameters $\gamma_\ell = J_\ell \langle d_\ell \rangle$.

Constrained free energy. We write the partition function—the normalization constant in the Boltzmann distribution over

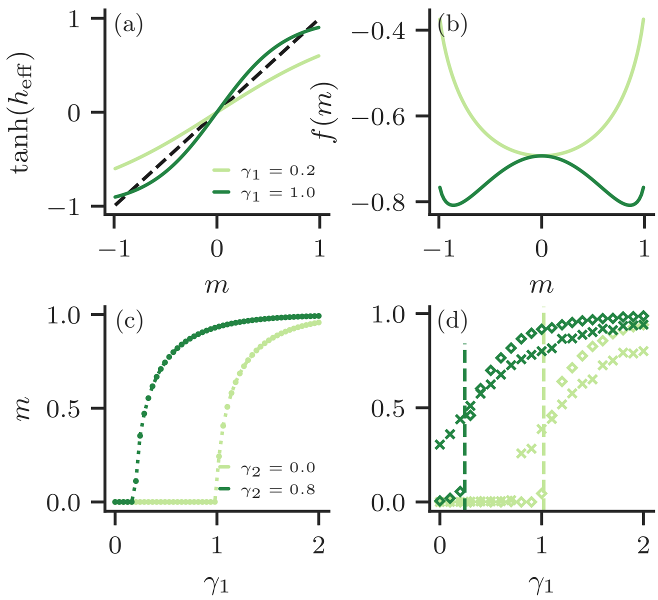

FIG. 2. (a) Shape of the right-hand side of Eq. (12) and (b) functional form of the mean-field constrained free energy density in the case in which $h = 0$, $\ell_{\max} = 2$, and $\gamma_2 = 0.5$. The dashed line in (a) is the bisector, corresponding to the left-hand side of the equation of state. (c) Phase transition in the $m(\gamma_1)$ phase space. (d) Numerical results for the absolute magnetization on random hypergraphs with $N = 1000$ nodes and generalized degrees $\langle d_1 \rangle = 20$, $\langle d_2 \rangle = 6$ (squares) and uncorrelated scale-free hypergraphs with $N = 1000$ and $\alpha_1 = \alpha_2 = 3$ (crosses). The dashed lines correspond to the critical pairwise coupling found in the mean-field approximation.

spin configurations $\{\mathbf{s}\}$—as:

$$Z = \sum_{\{\mathbf{s}\}} \exp[-\beta H(\mathbf{s})] \simeq \sum_{m} g(m) \exp[-\beta H(m)] \tag{10}$$

where $g(m)$ counts the number of configurations with magnetization m. Using Stirling's approximation, we can easily write $\log g(m)$ in the familiar form of a binary entropy. We can then introduce the constrained free energy density:

$$f(m) = \frac{H(m)}{N} - \frac{\log g(m)}{\beta N}. \tag{11}$$

The minimization of the constrained free energy reduces to the implicit equation:

$$m = \tanh\left[(\beta h_{\rm eff}(m))\right]. \tag{12}$$

Mean-field results with three-body interactions. When 3-body interactions are considered, the novel (CS) model displays significant differences in the solutions of the equation of state Eq. (12) and in the energy landscape with respect to the p-spin (BS) model [Figs. 2(a) and 2(b)], the solutions of the equation of state and the free-energy landscape of the BS model are presented in the SM). We see that the functional form of the constrained free energy $f(m)$ is symmetric for the CS model. As expected, in the CS model, the ferromagnetic ground state is degenerate *i.e.,* we have two symmetric minima corresponding to $\pm m$. In contrast, in the BS model, we have only one ferromagnetic solution with positive magnetization (see SM) due to the symmetry breaking induced by the coupling between an odd number of spins. We see in Fig. 2(c) that keeping fixed γ_2 and β, and varying γ_1 the system undergoes a continuous phase transition between the two phases mentioned above. This is not observed in the p-spin model, in which instead the introduction of many-body interactions ($\gamma_2 > 0$) gives an abrupt phase transition [31]. We have validated these results through Monte Carlo simulations [37] on random hypergraphs [38,39] [Fig. 2(d)]. When the degree and generalized degree distributions are sharply peaked around their average values (Erdős–Rényi-like hypergraphs) we have very good agreement between the mean-field analytics and the numerical results. This good agreement between mean-field theory and simulations is not always observed nor expected (e.g., in the 2D square lattice Ising model), random hypergraphs behave similarly—up to the effects introduced by the sparsity of the structure—to fully connected hypergraphs, where the mean-field theory is expected to be exact. When we consider heterogeneous structures—configuration hypergraphs in which power-laws with exponent $\alpha = 3$ describe the degree and generalized degree distributions—the fat-tailed degree distributions produce a lowering of the magnetization threshold. This lowering of the threshold for the onset of the phase transition echoes known results for pairwise dynamical models on scale-free networks, where the degree heterogeneity induces in finite systems small critical thresholds, which vanish as the system size is increased [25,40].

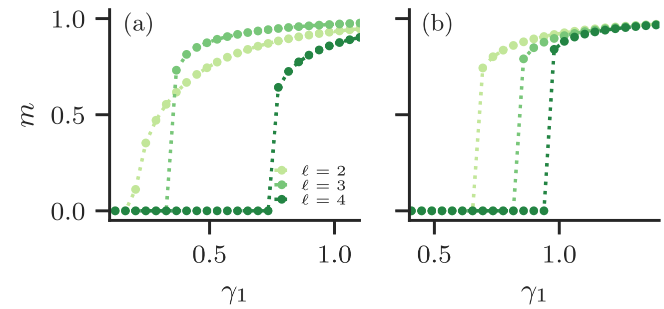

FIG. 3. Phase transition in the $m(\gamma_1)$ phase space for the mean-field solution of the higher-order Ising model (a) and for the ferromagnetic p-spin model (b) in the case with $h = 0$, pairwise interactions controlled by γ_1 and group interactions of order $\ell \in [2, 3, 4]$ with fixed $\gamma_\ell = 0.8$.

Phase transition beyond three-body interactions. In the case $\ell_{\max} = 2$, we have that the phase transition between the disordered and the magnetized phase is continuous, different from what is observed in $p = 3$-spin models, where the transition is explosive. This difference in the phase transition behavior between the two models vanishes once interactions of order $\ell \geqslant 3$ (4-body interactions and higher) are introduced (Fig. 3). The explosive transition results from the presence of powers of m in the expression for $h_{\rm eff}$. In the BS model, interactions of order ℓ introduce a factor of m^{ℓ}. Similarly, in the CS model, interactions of order $\ell > 2$ introduce powers of m in $h_{\rm eff}$. For example, considering a system with $\ell_{\max} = 3$ (4-body interactions), using Eqs. (8) and (9) we find the self-consistent equation for m a term proportional to m^2, triggering an abrupt phase transition when the pairwise coupling term is varied.

The physical impact of group interactions can be understood as follows. In the CS model, the unanimity rule (i.e., the fact that energy is minimized only when all the spins participating in a group are aligned) implies that, as group size increases ($\ell > 2$), unanimous groups become less frequent. However, when such unanimity does occur, it tends to align with the global magnetization. Consequently, when considering groups of size four or larger, two effects emerge: (i) the phase transition is hindered, with its onset shifting to higher values of the pairwise coupling γ_1; and (ii) when it eventually takes place, the nature of the phase transition changes from continuous to abrupt.

Beyond the mean-field approximation. To improve on our mean-field estimation, we can use high-temperature expansions of the free energy functional of our system at fixed order parameter [41–43]. We perform a Georges-Yedidia (G.-Y.) expansion by defining for a general spin system with Hamiltonian H, a magnetization-dependent free energy functional:

$$\mathcal{F}^{\beta}[\mathbf{m}] = \log \sum_{\{\mathbf{s}\}} \exp\left[-\beta H(\mathbf{s}) + \sum_i \rho_i^{\beta}(S_i - m_i)\right]. \quad (13)$$

The Lagrange multipliers ρ_i^{β} fix the magnetization at each site i to its thermal expectation. We then expand $\mathcal{F}^{\beta}[\mathbf{m}]$ around $\beta = 0$ using a Taylor expansion:

$$\mathcal{F}^{\beta}[\mathbf{m}] = \mathcal{F}^{\beta}[\mathbf{m}]|_{\beta=0} + \left.\frac{\partial \mathcal{F}^{\beta}[\mathbf{m}]}{\partial \beta}\right|_{\beta=0} \beta + \frac{1}{2} \left.\frac{\partial^2 \mathcal{F}^{\beta}[\mathbf{m}]}{\partial \beta^2}\right|_{\beta=0} \beta^2 + \cdots \quad (14)$$

The derivatives of $\mathcal{F}^{\beta}[\mathbf{m}]$ evaluated at $\beta = 0$ can be easily computed as spins are uncorrelated at infinite temperature: $\langle \prod_{i=1}^{\alpha} s_i \rangle|_{\beta=0} = \prod_{i=1}^{\alpha} m_i$. Computing the first two terms gives the standard mean-field theory, and higher-order derivatives provide corrections to the mean-field picture that are valid beyond the high-temperature regime. For example, for the Sherrington-Kirkpatrick model [44] the β^2 term gives the Onsager reaction term of the TAP equations [45]. We perform this expansion for the CS model in the simple cases of fully connected systems and d-regular 2-hypergraphs (see SM for the detailed derivation). In the fully connected case—as expected—the mean-field is exact in the thermodynamic limit: individual couplings vanish in the infinite system size and the expansion can be truncated after the second term. In the case of d-regular hypergraphs with only three-body interactions, we can write the constrained free-energy density as

$$f(m) = \log g(m) - \frac{\beta J_2 d}{2} m^2 - \frac{(\beta J_2)^2 d}{8}(1 - m^2)^2 + O(\beta^3). \quad (15)$$

We explore the effect of the first correction term on the critical temperature T_c and the spontaneous magnetization m_0—magnetization for $T \leqslant T_c$. To find the critical temperature, we expand Eq. (15) in powers of m and find the temperature for which the m^2 coefficient vanishes. This gives

$$T_c = J_2 d\left(1 - \frac{1}{2d} + O\left(\frac{1}{d^2}\right)\right) \quad (16)$$

revealing the role of sparsity in the hypergraph structure (encoded by the higher-order degree d), which produces a lowering of the critical temperature [see Fig. 4(a)]. For large values of d, we recover the fully connected regime where the mean-field predictions are exact. The spontaneous magnetization is obtained by expanding to order m^3 the minimization condition for $f(m)$. As we see in Fig. 4(b), the $1/d$ expansion that we performed varies the amplitude of the critical scaling $m \sim \sqrt{-t}$—where t is the reduced temperature $t = (T - T_c)/T_c$—but maintains the mean-field critical exponent.

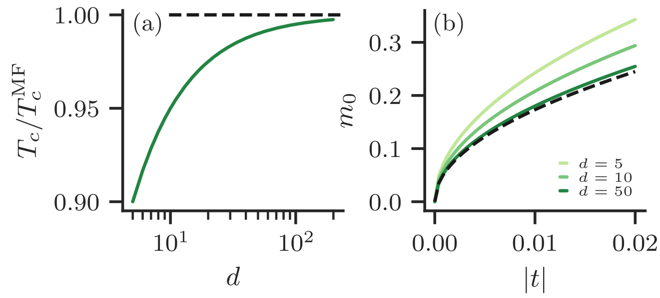

FIG. 4. (a) Ratio between the critical temperature obtained with the G.-Y. expansion and the mean-field critical temperature for d-regular 2-hypergraphs. (b) Results for the spontaneous magnetization in the vicinity of the critical point for d-regular 2-hypergraphs with $J_2 = 1$, t is the reduced temperature $t = (T - T_c)/T_c$. The dashed line is the mean-field result.

Discussion. We have introduced a novel Ising-like model on higher-order networks, with both arbitrary structure and orders of interactions. By an approximate mean-field solution, we have shown that, when 3-body interactions are introduced, the model displays the emergence of a continuous transition toward an ordered phase. This phenomenology is different from traditional p-spin models, which are characterized by an abrupt transition. Our results are confirmed by numerical simulations on homogeneous structures, while heterogeneous hypergraphs anticipate the onset of the transition. When further orders of interactions are considered, our model recovers explosive behavior. Finally, we performed a high-temperature expansion of the model, capturing corrections to the mean-field predictions of the behavior of the model by accounting for the effect of structure and the presence of feedback loops. Our work unveils new collective phenomena in spin models on complex interacting systems, revealing the importance of studying higher-order interactions beyond three-body.

Data availability. The data are not publicly available. The data and code to reproduce the results are available from the authors upon reasonable request.

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